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# Dynamical systems and sequence transformations

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**Abstract**
This paper discusses the connections between numerical methods for ordinary differential equations, fixed point iterations and sequence transformations.

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Books on chaos and fractals are concerned, on the one hand, with continuous dynamical systems, and, on the other hand, with the dynamics of iterations in discrete dynamical systems. There exists a strong connection between numerical methods for the integration of ordinary differential equations (ODEs) and fixed point iterations (FPIs). This connection has already been studied (see [14, chapter 5, pp 197–343] or [25, chapter 6, pp 165–201] for an introduction, and [30] for an extended review), but the dynamics of the iterations generated by a numerical method for ODEs was mainly used for understanding the behaviour of the solution of the differential equation.

In this paper, we are mostly interested in FPIs. We will see that FPIs can be considered as coming from methods for the numerical integration of ODEs. Reciprocally, numerical methods for ODEs can be used in the solution of fixed point problems. So, FPIs can lead to new methods for the numerical integration of ODEs, and conversely. The link with sequence transformations, used in numerical analysis to accelerate the convergence, will also be discussed.

## 1. Differential equations

We consider the system of $p$ autonomous differential equations

$$\dot{x}(t) = f(x(t)) \tag{1}$$

with the initial condition at 0 (which does not restrict the generality since $f$ is independent of $t$) $x(0) = x_0$. The *dot* represents differentiation with respect to $t$.

Let us recall some well known facts; see, e.g., [13], [15] or [31]. The path followed by the point $x(t)$ is called the *trajectory* of (1) since the variable $t$ can always be referred to as the *time*. The system (1) is called a *continuous dynamical system*. It is well known that its trajectory can exhibit quite different behaviours. *Transients* decay to zero when $t$ goes to infinity, while *steady states* remain after the transients disappear. If $\forall t \geqslant 0, x(t) = x^*$, then $\dot{x}(t) = f(x^*) = 0$. So, $x^*$ is a special type of steady state called an *equilibrium* point of the differential equation. Other important steady states are *orbits* (periodic trajectories such that $x(t) = x(t + nT)$, $\forall t \geqslant 0$ and $\forall n \in \mathbb{N}$, where $T$ is the *period*). These two types of steady states are the only possible ones for one- or two-dimensional systems but, as shown by Lorenz' discovery [26] in 1963, this is no longer true for higher-dimensional systems which can have a chaotic behaviour (see, e.g., [28]).

The equilibrium $x^*$ is called *stable* if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $\|x_0 - x^*\| < \delta$ implies $\|x(t) - x^*\| < \varepsilon$ for all $t > 0$. It is said to be *asymptotically stable* if it is stable and if $\exists \delta > 0$ such that $\|x_0 - x^*\| < \delta$ implies $\lim_{t\to\infty} x(t) = x^*$.

Assuming that $f$ is differentiable in a neighbourhood of $x^*$, we have (the *prime* denotes differentiation with respect to $x$)

$$f(x(t)) = f(x^* + x(t) - x^*) = f(x^*) + f'(x^*)(x(t) - x^*) + \mathrm{o}(x(t) - x^*)$$

and thus, setting $e(t) = x(t) - x^*$,

$$\dot{e}(t) = f'(x^*)e(t) + \mathrm{o}(e(t)). \tag{2}$$

If all eigenvalues of the Jacobian matrix $f'(x^*)$ have strictly negative real parts, then $x^*$ is asymptotically stable and, if $\|x_0 - x^*\| < \delta$,

$$\lim_{t\to\infty} x(t) = x^*.$$

## 2. Fixed point iterations

Let $F : \mathbb{R}^p \longmapsto \mathbb{R}^p$ and $x^*$ a fixed point of $F$, that is $x^* = F(x^*)$. We will discuss several types of FPIs for the computation of $x^*$ and show that they can be interpreted as methods for the numerical integration of ODEs. It means that FPIs can serve for integrating ODEs. Conversely, numerical methods for ODEs can be used for finding the fixed point $x^*$. This double connection can lead to new methods either for fixed point problems or for the numerical integration of ODEs. FPIs are also often referred to as *discrete dynamical systems*. For the stability of solutions of difference equations, see [11, 24, 33].

### *2.1. One-step iterations*

We consider the FPIs

$$x_{n+1} = F(x_n) \qquad n = 0, 1, \ldots \tag{3}$$

where $x_0 \in \mathbb{R}^p$. If $F$ is assumed to be differentiable in a neighbourhood of $x^*$, it is well known that $\exists V$ (called the *domain of attraction* of $x^*$ for the iterations (3)) such that, $\forall x_0 \in V$, the sequence $(x_n)$ converges to $x^*$ if the spectral radius of $F'(x^*)$ is strictly smaller than one (see, e.g., [27, p 300]). So, as pointed out in [9, 10, 23], the convergence of the iterative method (3) is related to the stability of the equilibrium point $x^*$. Indeed, the iterations (3) can be rewritten as

$$x_{n+1} = x_n + h(F(x_n) - x_n) \qquad n = 0, 1, \ldots \tag{4}$$

with $h = 1$, which is the Euler method with the stepsize $h = 1$ for the numerical integration of the system of autonomous differential equations

$$\dot{x}(t) = F(x(t)) - x(t) = f(x(t))$$

and $x_n$ is an approximation of the exact solution $x(t_n)$ of this differential equation, with the initial condition $x(0) = x_0$, at the point $t_n = nh$. Obviously, a different stepsize $h$ can be chosen since $x^*$ also satisfies $x^* = x^* + h(F(x^*) - x^*)$. So FPIs are related to numerical methods for the integration of ODEs.

The fixed point $x^*$ has to be an asymptotically stable equilibrium of the differential equation which, as seen in section 1, means that all eigenvalues of $f'(x^*)$ have strictly negative real parts. If they all have strictly positive real parts, one can consider, without loss of generality, the differential equation $\dot{x}(t) = f(x(t))$ with $f(x(t)) = x(t) - F(x(t))$.

If the Euler method (4) is applied with a stepsize $h \geqslant 0$, then the convergence condition given above for the FPIs is satisfied if the spectral radius of the matrix $I + h(F'(x^*) - I)$ is strictly smaller than one. This condition is equivalent to the condition that the eigenvalues of $F'(x^*)$ are inside the open disc of centre $1 - 1/h$ and radius $1/h$. Since $F'(x^*) - I = f'(x^*)$, this is also equivalent to the condition that the eigenvalues of $f'(x^*)$ are inside the open disc of centre $-1/h$ and radius $1/h$.

We remind ourselves that a numerical method for differential equations is *A-stable* if, when applied to the model problem $\dot{x}(t) = Ax(t)$, where $A$ is any matrix whose eigenvalues have negative real parts, then $\forall h > 0$, $\lim_{n\to\infty} x_n = 0$. Since $\lim_{t\to\infty} x(t) = 0$, it means that the exact and the approximate solutions must have the same asymptotic behaviour. A linear explicit method cannot be $A$-stable. The *domain of absolute stability* of a method is the domain of the complex plane where the eigenvalues of $hA$ have to be such that $\lim_{n\to\infty} x_n = 0$. For the Euler method, it is the open disc of centre $-1$ and radius 1. So, if we consider the linearized differential equation $\dot{x}(t) = f'(x^*)x(t)$, the domain of absolute stability of the Euler method coincides with the domain of attraction of $x^*$ for the iterations (4). As we will see now, this result is, in fact, more general.

Instead of the Euler method, other methods can be used for the numerical integration of (1). For example, we can take any explicit $r$-stage Runge–Kutta method:

$$x_{n+1} = x_n + h\,\varphi(x_n, h) \qquad n = 0, 1, \ldots \tag{5}$$

with

$$\begin{aligned} k_1(u, h) &= f(u) \\ k_2(u, h) &= f(u + a_{21}h\, k_1(u, h)) \\ &\vdots \\ k_r(u, h) &= f(u + a_{r1}h\, k_1(u, h) + \cdots + a_{r,r-1}h\, k_{r-1}(u, h)) \\ \varphi(u, h) &= c_1k_1(u, h) + \cdots + c_rk_r(u, h) \end{aligned}$$

with the consistency condition $c_1 + \cdots + c_r = 1$.

The sequence $(x_n)$ obtained by (5) can also be considered as FPIs for the computation of $x^*$ such that $f(x^*) = 0$.

We consider the case of one single equation, that is $p = 1$, and we set

$$\mathcal{S} = \{h \mid -1 < 1 + h\,\varphi'_x(x^*, h) < 1\}$$

where $\varphi'_x$ denotes the partial derivative of $\varphi$ with respect to its first variable. Then, $\forall h \in \mathcal{S}$, $\exists V(h)$ such that, $\forall x_0 \in V(h)$, the FPIs (5) converge to $x^*$.

Let us apply an explicit Runge–Kutta method to the model problem $\dot{x}(t) = f'(x^*)x(t)$. We have

**Theorem 1.** *The domain of absolute stability of an explicit Runge–Kutta method coincides with $\mathcal{S}$.*

**Proof.** The condition $-1 < 1 + h\,\varphi'_x(x^*, h) < 1$ gives (the $k_i$ are also derived with respect to their first variable)

$$-1 < 1 + h(c_1 k'_1(x^*, h) + \cdots + c_r k'_r(x^*, h)) < 1.$$

We have

$$\begin{aligned} k'_i(u, h) = {} & (1 + a_{i1} h\, k'_1(u, h) + \cdots + a_{i,i-1} h\, k'_{i-1}(u, h)) \\ & \times f'(u + a_{i1} h\, k_1(u, h) + \cdots + a_{i,i-1} h\, k_{i-1}(u, h)). \end{aligned}$$

But $k_i(x^*, h) = 0$ and, thus,

$$k'_i(x^*, h) = (1 + a_{i1} h\, k'_1(x^*, h) + \cdots + a_{i,i-1} h\, k'_{i-1}(x^*, h)) f'(x^*).$$

Moreover, it is easy to see by induction that $k'_i(x^*, h)x_n$ is identical to $k_i(x_n, h)$ obtained by an explicit Runge–Kutta method when applied to $\dot{x} = f'(x^*)x$. So, for this model problem,

$$x_{n+1} = x_n + h\varphi(x_n, h) = (1 + h\varphi'_x(x^*, h))x_n$$

and the domain of absolute stability of an explicit Runge–Kutta method is defined by $|1 + h\varphi'_x(x^*, h)| < 1$. □

This result was already implicitly used in [2].

Let us now discuss the choice of $h$ in the case $p = 1$. When an explicit $r$-stage Runge–Kutta method is applied to the model problem $\dot{x}(t) = \lambda x(t)$, we see, from the proof of theorem 1, that $x_{n+1} = P_r(h\lambda)x_n$ where $P_r$ is a polynomial of degree $r$. The domain of absolute stability is the set of $h\lambda$ such that $-1 < P_r(h\lambda) < 1$. In order to obtain a good approximation of $x^*$, the differential equation has to be integrated over a long time interval. So, $h$ must be as large as possible. The condition for absolute stability and convergence becomes $-2 < h\varphi'_x(x^*, h) < 0$ and the largest possible value for $h$ is such that $h\varphi'_x(x^*, h) = -2$. On the other hand, the error $|x_n - x(t_n)|$ increases with $h$ and, if $h$ is divided by $\alpha$, the error is divided by $\alpha^q$, where $q$ is the order of the Runge–Kutta method. However, in that case, the number of steps to reach the same abscissa is multiplied by $\alpha$. So, we have to make a compromise between these two antagonistic objectives. Moreover, the model problem is only an approximation of (2). So, taking $h$ such that $P_r(h\lambda) = 0$ (or, equivalently, so that $h\varphi'_x(x^*, h) = -1$) is the optimal choice. So, when a Runge–Kutta method is used for computing the fixed point $x^*$, we have, since, for all $h$, $\varphi(x^*, h) = 0$,

**Corollary 1.** *The FPIs $x_{n+1} = x_n + h\varphi(x_n, h)$, $n = 0, 1, \ldots$ have order 2 at least if and only if $h$ satisfies*

$$1 + h\varphi'_x(x^*, h) = P_r(hf'(x^*)) = 0.$$

So, accordingly, the optimal stepsize for the Euler method is $h = -1/f'(x^*)$, which leads to FPIs of order 2 at least for a simple zero.

Since $x^*$ is unknown, we will replace $h$ in (5) by a variable stepsize $h_n$ approximating the optimal one. So, we now consider iterations of the form

$$x_{n+1} = x_n + h_n\varphi(x_n, h_n) \qquad n = 0, 1, \ldots. \tag{6}$$

For these iterations, we have

$$\begin{aligned} x_{n+1} - x^* &= x_n - x^* + h_n[\varphi(x^*, h_n) + \varphi'_x(x^*, h_n)(x_n - x^*) + \mathcal{O}((x_n - x^*)^2)] \\ &= [1 + h_n\varphi'_x(x^*, h_n)](x_n - x^*) + h_n\mathcal{O}((x_n - x^*)^2) \end{aligned}$$

since $\forall h_n$, $\varphi(x^*, h_n) = 0$. So, we immediately obtain the following result which generalizes corollary 1

**Theorem 2.** *The FPIs (6) have a superlinear convergence (that is, an order strictly greater than 1) if and only if*

$$\lim_{n\to\infty} h_n\varphi'_x(x^*, h_n) = -1.$$

*If the expression for $h_n$ only involves $x_n$, then the order is 2 at least.*

The last statement comes from the fact that, in this case, the order is an integer.

Let us look into the explicit two-stage Runge–Kutta method. We have

$$h\varphi'_x(x^*, h) = (c_1 + c_2)hf'(x^*) + c_2a_{21}[hf'(x^*)]^2.$$

Since this method has order one at least, the consistency condition says that $c_1+c_2 = 1$. If $(h_n)$ tends to the optimal value $h = -1/f'(x^*)$, then $h\varphi'_x(x^*, h) = -1+c_2a_{21}$. So, the condition of theorem 2 cannot be satisfied unless $c_2$ and/or $a_{21}$ is zero. In both cases, the Euler method (4) is recovered. So, FPIs corresponding to an explicit $r$-stage Runge–Kutta method of order strictly greater than 1 cannot have a superlinear order of convergence. This also means that the order of the numerical method for integrating the ODE and the order of the corresponding FPI are not related.

This is the reason why we will now only consider the Euler method (4). Several choices for $h_n$ lead to FPIs (6) with a superlinear convergence

(1) If we take $h_n = -1/f'(x_n)$, the Euler method becomes

$$x_{n+1} = x_n - f(x_n)/f'(x_n) \qquad n = 0, 1, \ldots$$

and the Newton method for the solution of $f(x) = 0$ is recovered. It is well known that, for a simple zero, it has order 2.

(2) Since $x^*$ also satisfies $F(x^*) = F(F(x^*))$, the choice

$$h_n = \frac{1}{1 - \frac{F(F(x_n))-F(x_n)}{F(x_n)-x_n}} = -\frac{f(x_n)}{f(x_n + f(x_n)) - f(x_n)} \tag{7}$$

leads to the Steffensen method which has order 2 for a simple zero. We see that this method consists of approximating $f'(x_n)$ in the Newton method by $[f(x_n+f(x_n))-f(x_n)]/f(x_n)$.

(3) The secant method, whose order is $(1+\sqrt{5})/2 \simeq 1.618$ for a simple zero, is recovered by taking

$$h_n = \frac{1}{1 - \frac{F(x_n)-F(x_{n-1})}{x_n-x_{n-1}}} = -\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}.$$

The secant method is recovered by approximating $f'(x_n)$ in the Newton method by $[f(x_n) - f(x_{n-1})]/(x_n - x_{n-1})$.

(4) The choice

$$h_n = -\frac{f'(x_n)}{[f'(x_n)]^2 - f(x_n)f''(x_n)} = -\frac{1}{f'(x_n)}\,\frac{1}{1 - \frac{f(x_n)f''(x_n)}{[f'(x_n)]^2}}$$

gives a method which has order 2 even for a multiple zero.

(5) The choice

$$h_n = -\frac{2f'(x_n)}{2[f'(x_n)]^2 - f(x_n)f''(x_n)}$$
$$= -\frac{1}{f'(x_n)} \frac{1}{1 - \frac{f(x_n)f''(x_n)}{2[f'(x_n)]^2}}$$

leads to the Halley method which has order 3 for a simple zero.

(6) Taking

$$h_n = -\frac{1}{f'(x_n)}\left[1 + \frac{f(x_n)f''(x_n)}{2[f'(x_n)]^2}\right]$$

gives another method of order 3 for a simple zero. It is attributed to Chebyshev (see [7]).

In these last three methods, the first and/or the second derivatives can be replaced by finite difference approximations, thus leading to other methods. Other choices of $h_n$ can also be of interest.

Explicit multistep, as well as prediction–correction, methods can also be considered. All these methods can be used with a variable stepsize. However, as noticed in [16], these methods can introduce spurious fixed points and the order of convergence of the related FPIs does not seem to be influenced by the order of the method for ODEs. Moreover, none of them is $A$-stable since they are linear and explicit. So, an explicit $A$-stable method should be nonlinear as the method proposed in [1] (see also [34]) which is

$$x_{n+1} = x_n + h \frac{2f(x_n)}{2f(x_n) - hf'(x_n)} f(x_n) \qquad n = 0, 1, \ldots.$$

This method for differential equations was obtained from the confluent form of the $\rho$ algorithm, a procedure for the computation of the limit of a function when the variable tends to infinity. It is strongly related to the $\rho$ algorithm, a sequence transformation in the sense defined in section 3; on this topic, see [4]. This method can also be recovered via the Padé approximation as explained in [6, pp 238–41]. It is $A$-stable and has order 2. It can also be interpreted as the Euler method with variable stepsize

$$h_n = h \frac{2f(x_n)}{2f(x_n) - hf'(x_n)}$$

applied to $\dot{x}(t) = f(x(t))$. Replacing $f'(x_n)$ by its approximation $(f(x_n) - f(x_{n-1}))/(x_n - x_{n-1})$ leads to the Euler method with variable stepsize

$$h_n = h \frac{2f(x_n)}{2f(x_n) - h\frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}}.$$

### 2.2. *Steffensen-type iterations*

We set $F_0(x) = x$ and $F_{i+1}(x) = F(F_i(x))$ for $i \geqslant 0$. A Steffensen-type method for the computation of the fixed point $x^*$ consists of the iterations

$$x_{n+1} = G(x_n, F_1(x_n), \ldots, F_k(x_n)) \qquad n = 0, 1, \ldots \tag{8}$$

where $k$ is a fixed integer. Obviously, the Steffensen method described in section 2.1 falls into this category. Steffensen-type methods are given, for example, in [8, 17, 20–22].

As in section 2.1, the iterations (8) can be interpreted as the Euler method with stepsize $h = 1$ applied to the differential equation

$$\dot{x}(t) = G(x(t), F_1(x(t)), \ldots, F_k(x(t))) - x(t) \tag{9}$$

that is

$$x_{n+1} = x_n + h[G(x_n, F_1(x_n), \ldots, F_k(x_n)) - x_n] \qquad n = 0, 1, \ldots. \tag{10}$$

For example, in the case of the Steffensen method, we have

$$\dot{x}(t) = -\frac{(F_1(x(t)) - x(t))^2}{F_2(x(t)) - 2F_1(x(t)) + x(t)}.$$

This differential equation can be integrated by the Euler method with one of the variable stepsizes given above or by any other numerical method.

Let us assume that the function $G$ is *translative*, which means that

$$G(u_0 + b, \ldots, u_k + b) = G(u_0, \ldots, u_k) + b$$

and *homogeneous*, that is

$$G(au_0, \ldots, au_k) = aG(u_0, \ldots, u_k).$$

A function $G$ which is translative and homogeneous is called *quasi-linear*.

By the translativity property, we have

$$x_{n+1} - x^* = G(x_n - x^*, F_1(x_n) - x^*, \ldots, F_k(x_n) - x^*).$$

Since $G$ is homogeneous, it holds that

$$x_{n+1} - x^* = (x_n - x^*)\; G(1, (F_1(x_n) - x^*)/(x_n - x^*), \ldots, (F_k(x_n) - x^*)/(x_n - x^*)).$$

Let us set, for simplicity, $F'(x^*) = \rho \neq 1$. We have $F_i(x_n) - x^* = \rho^i(x_n - x^*) + \mathrm{o}(x_n - x^*)$ and it follows from what precedes [3] (compare with [32, theorem 1, p 113])

**Theorem 3.** *If* $G(1, \rho, \ldots, \rho^k) = 0$*, then the sequence* $(x_n)$ *given by (8) converges superlinearly.*

As we will see in section 3, Steffensen-type iterations are linked to sequence transformations and this result is related to theorems 7 and 8 of section 3. This connection has been studied in [32, pp 112ff].

Let us come back to the differential equation (9). Since $f(u) = F(u) - u$, it is easy to see that

$$F_i(u) = u + \varphi_i(u) \qquad i = 0, \ldots, k$$

with

$$\varphi_0(u) = 0$$
$$\varphi_i(u) = \varphi_{i-1}(u) + f(F_{i-1}(u)) \qquad i = 1, \ldots, k.$$

Thanks to the translativity property of $G$, we have

$$\begin{aligned} G(u, F_1(u), \ldots, F_k(u)) &= G(u + \varphi_0(u), \ldots, u + \varphi_k(u)) \\ &= u + G(\varphi_0(u), \ldots, \varphi_k(u)) \end{aligned}$$

and the differential equation (9) becomes

$$\dot{x}(t) = G(\varphi_0(x(t)), \ldots, \varphi_k(x(t)))$$

while the FPIs (10) change into

$$x_{n+1} = x_n + h\; G(\varphi_0(x_n), \ldots, \varphi_k(x_n)).$$

*2.3. Multistep iterations*

General fixed point multistep iterations for $x^* = G(x^*, \ldots, x^*)$ have the form

$$x_{n+1} = G(x_n, \ldots, x_{n-k}) \qquad n = k, k+1, \ldots \tag{11}$$

where $k$ is a fixed integer and where $x_0, \ldots, x_k$ are arbitrary initial vectors. In the case $p = 1$, the secant method given in section 2.1 also belongs to this class.

We can write these iterations as

$$x_{n+1} = x_n + h[G(x_n, \ldots, x_{n-k}) - x_n]$$

with $h = 1$. So, they can be viewed as the Euler method with stepsize $h = 1$ for the delay differential equation

$$\dot{x}(t) = G(x(t), x(t-h), \ldots, x(t-kh)) - x(t).$$

In the case of the secant method, we obtain

$$\dot{x}(t) = -\frac{x(t) - x(t-h)}{f(x(t)) - f(x(t-h))} f(x(t)).$$

Instead of $h = 1$, a variable stepsize can be chosen, as in section 2.1, or another numerical method for the integration of this delay differential equation.

Let us consider the particular case of fixed point multistep iterations of the form

$$x_{n+1} = \sum_{i=0}^{k} (h\beta_i f(x_{n+i-k}) - \alpha_i x_{n+i-k}) \qquad n = k, k+1, \ldots \tag{12}$$

with $h = 1$ and $x_0, \ldots, x_k$ given. They can be considered as being produced by a multistep method for the numerical integration of the differential equation $\dot{x}(t) = f(x(t))$. Let $x^*$ be a zero of $f$. It holds that

$$x^* = \sum_{i=0}^{k} (h\beta_i f(x^*) - \alpha_i x^*) = -x^* \sum_{i=0}^{k} \alpha_i.$$

So, we must have $1 + \sum_{i=0}^{k} \alpha_i = 0$, which is one of the necessary and sufficient conditions for the method (12) to be consistent with the differential equation. The initializations $x_0, \ldots, x_k$ must be obtained by a one-step method for integrating the differential equation. Again, the method can be used with a variable stepsize.

## 3. Sequence transformations

It is well known that fixed point methods are related to sequence transformations (used for accelerating the convergence of sequences by an extrapolation procedure, see [5]). So, we will begin with some known results about such transformations.

Let $(S_n)$ be a scalar sequence converging to a limit $S$. A sequence transformation $T$ is a mapping of $(S_n)$ into the new sequence $(T_n)$ given by

$$T_n = G(S_n, \ldots, S_{n+k}) \qquad n = 0, 1, \ldots \tag{13}$$

where the function $G$ is assumed to be translative (see section 2.2 for the definition).

As proved in [3], we have

**Theorem 4.** *A necessary and sufficient condition for $G$ to be translative is that it has the form $G(u_0, \ldots, u_k) = g(u_0, \ldots, u_k)/Dg(u_0, \ldots, u_k)$, where $D$ is the differential operator $D = \partial/\partial u_0 + \cdots + \partial/\partial u_k$ and where the function $g$ satisfies $D^2 g \equiv 0$. An equivalent condition is $DG \equiv 1$.*

For example, the function $g$ corresponding to the Aitken $\Delta^2$ process is $g(u_0, u_1, u_2) = u_0u_2 - u_1^2$.

We have [3]

**Theorem 5.** *If G is translative then*

$$G(u_0, \ldots, u_k) = \sum_{i=0}^{k} u_i \frac{\partial G}{\partial u_i}(u_0, \ldots, u_k).$$

Let us assume that $G$ is quasi-linear (that is, translative and homogeneous). Since $G$ is translative, we have

$$G(u_0, \ldots, u_k) = u_0 + G(0, u_1 - u_0, \ldots, u_k - u_0).$$

But $u_i - u_0 = (u_i - u_{i-1}) + (u_{i-1} - u_{i-2}) + \cdots + (u_1 - u_0)$ and, so,

$$G(u_0, \ldots, u_k) = u_0 + K(u_1 - u_0, \ldots, u_k - u_{k-1}).$$

Since $G$ is homogeneous, so is $K$ and it follows that

$$\begin{aligned} G(u_0, \ldots, u_k) &= u_0 + (u_1 - u_0)\, K\left(1, \frac{u_2 - u_1}{u_1 - u_0}, \ldots, \frac{u_k - u_{k-1}}{u_1 - u_0}\right) \\ &= u_0 + (u_1 - u_0)\, H\left(\frac{u_2 - u_1}{u_1 - u_0}, \ldots, \frac{u_k - u_{k-1}}{u_{k-1} - u_{k-2}}\right) \qquad (14) \\ &= u_0 + (u_1 - u_0)\, G\left(0, 1, \frac{u_2 - u_0}{u_1 - u_0}, \ldots, \frac{u_k - u_0}{u_1 - u_0}\right). \qquad (15) \end{aligned}$$

A sequence $(S_n)$ which converges to $S$ and is such that $\exists \rho \neq 1$, $\lim_{n\to\infty}(S_{n+1} - S)/(S_n - S) = \rho$ is called *linearly convergent*. We say that $T$ accelerates the convergence of $(S_n)$ (or, equivalently, that $(T_n)$ converges faster than $(S_n)$) if $\lim_{n\to\infty}(T_n - S)/(S_n - S) = 0$. The acceleration of convergence of linearly convergent sequences is, as proved by Germain–Bonne [12], given by a property of the function $H$

**Theorem 6.** *A necessary and sufficient condition that T accelerates the convergence of linearly convergent sequences is that* $H(\rho, \ldots, \rho) = 1/(1 - \rho)$.

We also have [3]

**Theorem 7.** *Let* $(S_n)$ *be a linearly convergent sequence. If* $Dg(1, \rho, \ldots, \rho^k) \neq 0$, *and if* $|g(1, \rho, \ldots, \rho^k)| < M$, *then* $(T_n)$ *converges to S. Moreover, if* $g(1, \rho, \ldots, \rho^k) = G(1, \rho, \ldots, \rho^k) = 0$ *then* $(T_n)$ *converges faster than* $(S_n)$.

Let us now relate sequence transformations of the form (13) to FPIs. For computing a fixed point of $F$, we associate the Steffensen-type iterative method (8) with the transformation $T$ defined by (13). For example, the Steffensen method is associated with the Aitken $\Delta^2$ process which is given by

$$T_n = S_n - \frac{(S_{n+1} - S_n)^2}{S_{n+2} - 2S_{n+1} + S_n} \qquad n = 0, 1, \ldots.$$

We have the following result which relates the property of convergence acceleration of linearly convergent sequences with a translative transformation $T$ and the superlinear convergence of the associated Steffensen-type FPIs [3]

**Theorem 8.** *Let* $(S_n)$ *be a linearly convergent sequence and let* $x^*$ *be a fixed point of F such that* $F'(x^*) \neq 1$. *A necessary and sufficient condition that* $\lim_{n\to\infty}(T_n - S)/(S_n - S) = 0$ *is that* $\lim_{n\to\infty}(x_{n+1} - x^*)/(x_n - x^*) = 0$.

Theorems 7 and 8 are related to theorem 3 of section 2.2.

From (14) and (15), we see that the iterations (8) can also be interpreted as the Euler method for $\dot{x}(t) = F(x(t)) - x(t)$ with variable stepsize

$$h_n = H\left(\frac{F_2(x_n) - F_1(x_n)}{F_1(x_n) - F_0(x_n)}, \ldots, \frac{F_k(x_n) - F_{k-1}(x_n)}{F_{k-1}(x_n) - F_{k-2}(x_n)}\right)$$
$$= G\left(0, 1, \frac{F_2(x_n) - x_n}{F_1(x_n) - x_n}, \ldots, \frac{F_k(x_n) - x_n}{F_1(x_n) - x_n}\right).$$

These results, which have been extended to the vector case in [18, 29], complete those of section 2.2. In particular, for the Steffensen method, formula (7) is recovered.

## 4. Conclusion

In section 2, we saw that FPIs can be seen as methods for the numerical integration of ODEs with a variable stepsize. So, any fixed point method leads to a numerical method for ODEs. Conversely, any method for ODEs gives rise to fixed point iterations.

In section 3, we saw that a sequence transformation can be turned into a fixed point method by replacing $S_{n+i}$ by $F_i(S_n)$ and vice versa. So, thanks to the discussion of section 2, any sequence transformation is also related to a numerical method for the integration of the autonomous differential equation (1), where $f(u) = F(u) - u$.

In conclusion, methods for the numerical integration of initial value problems for ordinary differential equations, fixed point iterations, and sequence transformations have been connected.

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